# Challenge Problems

## Chapter 1

A Click here for answers. S Click here for solutions.

**1.** Draw the graph of the equation $|x| + |y| = 1 + |xy|$.

**2.** Draw the graph of the equation $x^2y - y^3 - 5x^2 + 5y^2 = 0$ without making a table of values.

## Chapter 3

A Click here for answers. S Click here for solutions.

**1.** (a) Find the domain of the function $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}$.
(b) Find $f'(x)$.
(c) Check your work in parts (a) and (b) by graphing $f$ and $f'$ on the same screen.

**2.** Find the $n$th derivative of the function $f(x) = x^n/(1 - x)$.

**3.** Prove that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

**4.** Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin(3 + x)^2 - \sin 9}{x}$.

**5.** Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.

## Chapter 4

A Click here for answers. S Click here for solutions.

**1.** Find the absolute maximum value of the function

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - 2|}$$

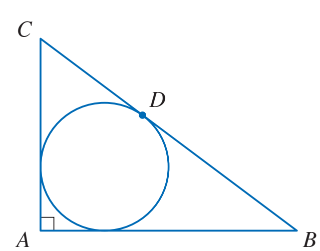
**2.** Find a function $f$ such that $f'(-1) = \frac{1}{2}$, $f'(0) = 0$, and $f''(x) > 0$ for all $x$, or prove that such a function cannot exist.

**3.** A triangle with sides $a$, $b$, and $c$ varies with time $t$, but its area never changes. Let $\theta$ be the angle opposite the side of length $a$ and suppose $\theta$ always remains acute.
(a) Express $d\theta/dt$ in terms of $b$, $c$, $\theta$, $db/dt$, and $dc/dt$.
(b) Express $da/dt$ in terms of the quantities in part (a).

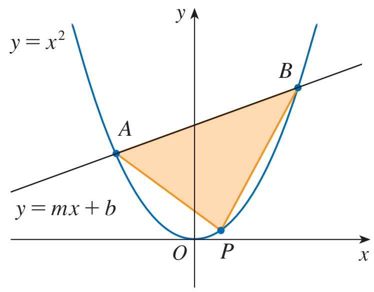
**4.** (a) Let $ABC$ be a triangle with right angle $A$ and hypotenuse $a = |BC|$. (See the figure.) If the inscribed circle touches the hypotenuse at $D$, show that

$$|CD| = \tfrac{1}{2}(|BC| + |AC| - |AB|)$$

(b) If $\theta = \frac{1}{2}\angle C$, express the radius $r$ of the inscribed circle in terms of $a$ and $\theta$.
(c) If $a$ is fixed and $\theta$ varies, find the maximum value of $r$.



**5.** The line $y = mx + b$ intersects the parabola $y = x^2$ in points $A$ and $B$ (see the figure). Find the point $P$ on the arc $AOB$ of the parabola that maximizes the area of the triangle $PAB$.




## Chapter 5

A Click here for answers.

S Click here for solutions.

**1.** Show that $\dfrac{1}{17} \leqslant \displaystyle\int_1^2 \frac{1}{1 + x^4}\,dx \leqslant \frac{7}{24}$.

**2.** If $n$ is a positive integer, prove that

$$\int_0^1 (\ln x)^n\,dx = (-1)^n n!$$

**3.** (a) Evaluate $\int_0^n [\![x]\!]\,dx$, where $n$ is a positive integer.

(b) Evaluate $\int_a^b [\![x]\!]\,dx$, where $a$ and $b$ are real numbers with $0 \leqslant a < b$.

**4.** Suppose that $f$ is a positive function such that $f'$ is continuous.

(a) How is the graph of $y = f(x)\sin nx$ related to the graph of $y = f(x)$? What happens as $n \to \infty$?

(b) Make a guess as to the value of the limit

$$\lim_{n\to\infty} \int_0^1 f(x)\sin nx\,dx$$

based on graphs of the integrand.

(c) Using integration by parts, confirm the guess that you made in part (b). [Use the fact that, since $f'$ is continuous, there is a constant $M$ such that $|f'(x)| \leqslant M$ for $0 \leqslant x \leqslant 1$.]

**5.** Prove that if $f$ is continuous, then $\displaystyle\int_0^x f(u)(x - u)\,du = \int_0^x \left(\int_0^u f(t)\,dt\right) du$.

**6.** A rocket is fired straight up, burning fuel at the constant rate of $b$ kilograms per second. Let $v = v(t)$ be the velocity of the rocket at time $t$ and suppose that the velocity $u$ of the exhaust gas is constant. Let $M = M(t)$ be the mass of the rocket at time $t$ and note that $M$ decreases as the fuel burns. If we neglect air resistance, it follows from Newton's Second Law that

$$F = M\frac{dv}{dt} - ub$$

where the force $F = -Mg$. Thus

$$\boxed{1} \qquad M\frac{dv}{dt} - ub = -Mg$$

Let $M_1$ be the mass of the rocket without fuel, $M_2$ the initial mass of the fuel, and $M_0 = M_1 + M_2$. Then, until the fuel runs out at time $t = M_2 b$, the mass is $M = M_0 - bt$.

(a) Substitute $M = M_0 - bt$ into Equation 1 and solve the resulting equation for $v$. Use the initial condition $v(0) = 0$ to evaluate the constant.

(b) Determine the velocity of the rocket at time $t = M_2/b$. This is called the *burnout velocity*.

(c) Determine the height of the rocket $y = y(t)$ at the burnout time.

(d) Find the height of the rocket at any time $t$.

**7.** If $0 < a < b$, find $\lim_{t \to 0} \left\{ \int_0^1 [bx + a(1 - x)]^t \, dx \right\}^{1/t}$.

**8.** The **Chebyshev polynomials** $T_n$ are defined by

$$T_n(x) = \cos(n \arccos x) \qquad n = 0, 1, 2, 3, \ldots$$

(a) What are the domain and range of these functions?
(b) We know that $T_0(x) = 1$ and $T_1(x) = x$. Express $T_2$ explicitly as a quadratic polynomial and $T_3$ as a cubic polynomial.
(c) Show that, for $n \geqslant 1$,

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$$

(d) Use part (c) to show that $T_n$ is a polynomial of degree $n$.
(e) Use parts (b) and (c) to express $T_4$, $T_5$, $T_6$, and $T_7$ explicitly as polynomials.
(f) What are the zeros of $T_n$? At what numbers does $T_n$ have local maximum and minimum values?
(g) Graph $T_2$, $T_3$, $T_4$, and $T_5$ on a common screen.
(h) Graph $T_5$, $T_6$, and $T_7$ on a common screen.
(i) Based on your observations from parts (g) and (h), how are the zeros of $T_n$ related to the zeros of $T_{n+1}$? What about the $x$-coordinates of the maximum and minimum values?
(j) Based on your graphs in parts (g) and (h), what can you say about $\int_{-1}^{1} T_n(x) \, dx$ when $n$ is odd and when $n$ is even?
(k) Use the substitution $u = \arccos x$ to evaluate the integral in part (j).
(l) The family of functions $f(x) = \cos(c \arccos x)$ are defined even when $c$ is not an integer (but then $f$ is not a polynomial). Describe how the graph of $f$ changes as $c$ increases.

**9.** Evaluate $\lim_{n \to \infty} \left( \dfrac{1}{\sqrt{n}\sqrt{n+1}} + \dfrac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \dfrac{1}{\sqrt{n}\sqrt{n+n}} \right)$.

## Chapter 6

A Click here for answers.

S Click here for solutions.

**1.** Find the area of the region $S = \{(x, y) \mid x \geqslant 0, \ y \leqslant 1, \ x^2 + y^2 \leqslant 4y\}$.

**2.** There is a line through the origin that divides the region bounded by the parabola $y = x - x^2$ and the $x$-axis into two regions with equal area. What is the slope of that line?

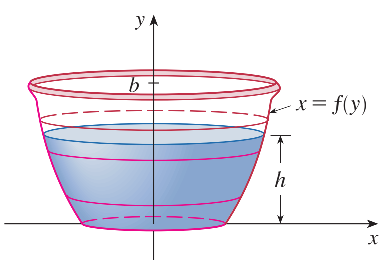



**FIGURE FOR PROBLEM 3**

**3.** A *clepsydra*, or water clock, is a glass container with a small hole in the bottom through which water can flow. The "clock" is calibrated for measuring time by placing markings on the container corresponding to water levels at equally spaced times. Let $x = f(y)$ be continuous on the interval $[0, b]$ and assume that the container is formed by rotating the graph of $f$ about the $y$-axis. Let $V$ denote the volume of water and $h$ the height of the water level at time $t$.
(a) Determine $V$ as a function of $h$.
(b) Show that

$$\frac{dV}{dt} = \pi [f(h)]^2 \frac{dh}{dt}$$

(c) Suppose that $A$ is the area of the hole in the bottom of the container. It follows from Torricelli's Law that the rate of change of the volume of the water is given by

$$\frac{dV}{dt} = kA\sqrt{h}$$

where $k$ is a negative constant. Determine a formula for the function $f$ such that $dh/dt$ is a constant $C$. What is the advantage in having $dh/dt = C$?

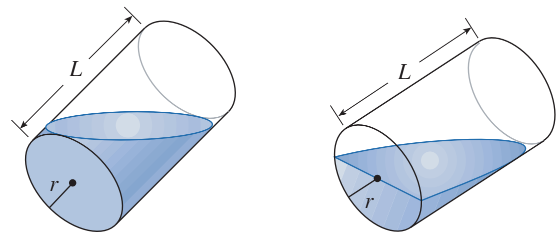
**4.** A cylindrical glass of radius $r$ and height $L$ is filled with water and then tilted until the water remaining in the glass exactly covers its base.
(a) Determine a way to "slice" the water into parallel rectangular cross-sections and then *set up* a definite integral for the volume of the water in the glass.

(b) Determine a way to "slice" the water into parallel cross-sections that are trapezoids and then *set up* a definite integral for the volume of the water.

(c) Find the volume of water in the glass by evaluating one of the integrals in part (a) or part (b).

(d) Find the volume of the water in the glass from purely geometric considerations.

(e) Suppose the glass is tilted until the water exactly covers half the base. In what direction can you "slice" the water into triangular cross-sections? Rectangular cross-sections? Cross-sections that are segments of circles? Find the volume of water in the glass.




**5.** Suppose that the density of seawater, $\rho = \rho(z)$, varies with the depth $z$ below the surface.

(a) Show that the hydrostatic pressure is governed by the differential equation

$$\frac{dP}{dz} = \rho(z)g$$

where $g$ is the acceleration due to gravity. Let $P_0$ and $\rho_0$ be the pressure and density at $z = 0$. Express the pressure at depth $z$ as an integral.

(b) Suppose the density of seawater at depth $z$ is given by $\rho = \rho_0 e^{z/H}$ where $H$ is a positive constant. Find the total force, expressed as an integral, exerted on a vertical circular porthole of radius $r$ whose center is at a distance $L > r$ below the surface.

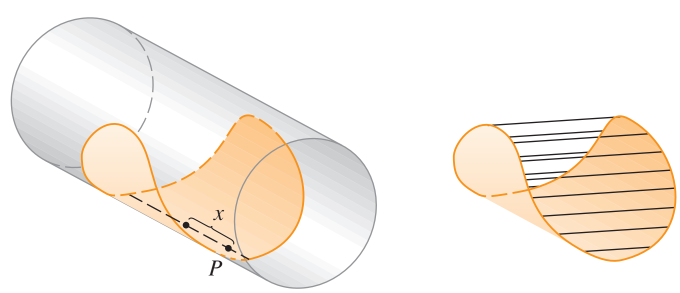
CAS **6.** Suppose we are planning to make a taco from a round tortilla with diameter 8 inches by bending the tortilla so that it is shaped as if it is partially wrapped around a circular cylinder. We will fill the tortilla to the edge (but no more) with meat, cheese, and other ingredients. Our problem is to decide how to curve the tortilla in order to maximize the volume of food it can hold.

(a) We start by placing a circular cylinder of radius $r$ along a diameter of the tortilla and folding the tortilla around the cylinder. Let $x$ represent the distance from the center of the tortilla to a point $P$ on the diameter (see the figure). Show that the cross-sectional area of the filled taco in the plane through $P$ perpendicular to the axis of the cylinder is

$$A(x) = r\sqrt{16 - x^2} - \tfrac{1}{2}r^2 \sin\left(\frac{2}{r}\sqrt{16 - x^2}\right)$$

and write an expression for the volume of the filled taco.

(b) Determine (approximately) the value of $r$ that maximizes the volume of the taco. (Use a graphical approach with your CAS.)




**7.** In a famous 18th-century problem, known as *Buffon's needle problem*, a needle of length $h$ is dropped onto a flat surface (for example, a table) on which parallel lines $L$ units apart, $L \geqslant h$, have been drawn. The problem is to determine the probability that the needle will come to rest intersecting one of the lines. Assume that the lines run east-west, parallel to the $x$-axis in a rectangular

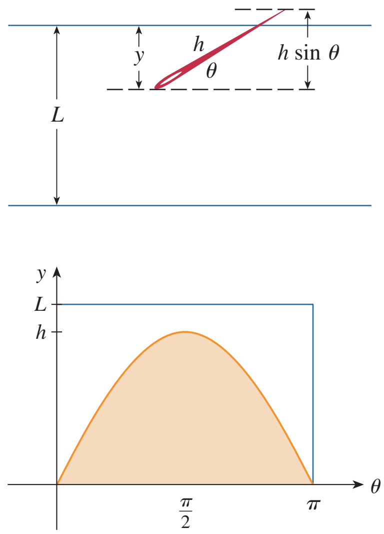



FIGURE FOR PROBLEM 7

coordinate system (as in the figure). Let $y$ be the distance from the "southern" end of the needle to the nearest line to the north. (If the needle's southern end lies on a line, let $y = 0$. If the needle happens to lie east-west, let the "western" end be the "southern" end.) Let $\theta$ be the angle that the needle makes with a ray extending eastward from the "southern" end. Then $0 \leqslant y \leqslant L$ and $0 \leqslant \theta \leqslant \pi$. Note that the needle intersects one of the lines only when $y < h \sin \theta$. Now, the total set of possibilities for the needle can be identified with the rectangular region $0 \leqslant y \leqslant L$, $0 \leqslant \theta \leqslant \pi$, and the proportion of times that the needle intersects a line is the ratio

$$\frac{\text{area under } y = h \sin \theta}{\text{area of rectangle}}$$

This ratio is the probability that the needle intersects a line. Find the probability that the needle will intersect a line if $h = L$. What if $h = L/2$?

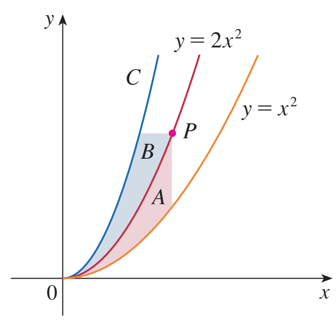
**8.** If the needle in Problem 7 has length $h > L$, it's possible for the needle to intersect more than one line.

(a) If $L = 4$, find the probability that a needle of length 7 will intersect at least one line. [*Hint:* We can proceed as in Problem 7. Define $y$ as before; then the total set of possibilities for the needle can be identified with the same rectangular region $0 \leqslant y \leqslant L$, $0 \leqslant \theta \leqslant \pi$. What portion of the rectangle corresponds to the needle intersecting a line?]

(b) If $L = 4$, find the probability that a needle of length 7 will intersect *two* lines.

(c) If $2L < h \leqslant 3L$, find a general formula for the probability that the needle intersects three lines.

**9.** The figure shows a curve $C$ with the property that, for every point $P$ on the middle curve $y = 2x^2$, the areas $A$ and $B$ are equal. Find an equation for $C$.



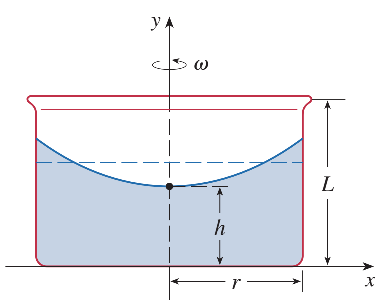




FIGURE FOR PROBLEM 10

**10.** A cylindrical container of radius $r$ and height $L$ is partially filled with a liquid whose volume is $V$. If the container is rotated about its axis of symmetry with constant angular speed $\omega$, then the container will induce a rotational motion in the liquid around the same axis. Eventually, the liquid will be rotating at the same angular speed as the container. The surface of the liquid will be convex, as indicated in the figure, because the centrifugal force on the liquid particles increases with the distance from the axis of the container. It can be shown that the surface of the liquid is a paraboloid of revolution generated by rotating the parabola

$$y = h + \frac{\omega^2 x^2}{2g}$$

about the $y$-axis, where $g$ is the acceleration due to gravity.

(a) Determine $h$ as a function of $\omega$.

(b) At what angular speed will the surface of the liquid touch the bottom? At what speed will it spill over the top?

(c) Suppose the radius of the container is 2 ft, the height is 7 ft, and the container and liquid are rotating at the same constant angular speed. The surface of the liquid is 5 ft below the top of the tank at the central axis and 4 ft below the top of the tank 1 ft out from the central axis.

(i) Determine the angular speed of the container and the volume of the fluid.

(ii) How far below the top of the tank is the liquid at the wall of the container?

**11.** If the tangent at a point $P$ on the curve $y = x^3$ intersects the curve again at $Q$, let $A$ be the area of the region bounded by the curve and the line segment $PQ$. Let $B$ be the area of the region defined in the same way starting with $Q$ instead of $P$. What is the relationship between $A$ and $B$?

## Chapter 8

A Click here for answers. S Click here for solutions.

**1.** (a) Show that $\tan \frac{1}{2}x = \cot \frac{1}{2}x - 2\cot x$.

(b) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n}$$

**2.** A function $f$ is defined by

$$f(x) = \lim_{n\to\infty} \frac{x^{2n} - 1}{x^{2n} + 1}$$

Where is $f$ continuous?

**3.** (a) Show that for $xy \neq -1$,

$$\arctan x - \arctan y = \arctan \frac{x - y}{1 + xy}$$

if the left side lies between $-\pi/2$ and $\pi/2$.

(b) Show that

$$\arctan \tfrac{120}{119} - \arctan \tfrac{1}{239} = \frac{\pi}{4}$$

(c) Deduce the following formula of John Machin (1680–1751):

$$4 \arctan \tfrac{1}{5} - \arctan \tfrac{1}{239} = \frac{\pi}{4}$$

(d) Use the Maclaurin series for arctan to show that

$$0.197395560 < \arctan \tfrac{1}{5} < 0.197395562$$

(e) Show that

$$0.004184075 < \arctan \tfrac{1}{239} < 0.004184077$$

(f) Deduce that, correct to seven decimal places,

$$\pi \approx 3.1415927$$

Machin used this method in 1706 to find $\pi$ correct to 100 decimal places. Recently, with the aid of computers, the value of $\pi$ has been computed to increasingly greater accuracy. In 1999, Takahashi and Kanada, using methods of Borwein and Brent/Salamin, calculated the value of $\pi$ to 206,158,430,000 decimal places!

**4.** (a) Prove a formula similar to the one in Problem 3(a) but involving arccot instead of arctan.

(b) Find the sum of the series

$$\sum_{n=0}^{\infty} \operatorname{arccot}(n^2 + n + 1)$$

**5.** Find the interval of convergence of $\sum_{n=1}^{\infty} n^3 x^n$ and find its sum.

**6.** If $a_0 + a_1 + a_2 + \cdots + a_k = 0$, show that

$$\lim_{n\to\infty} \left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right) = 0$$

If you don't see how to prove this, try the problem-solving strategy of *using analogy* (see page 86). Try the special cases $k = 1$ and $k = 2$ first. If you can see how to prove the assertion for these cases, then you will probably see how to prove it in general.

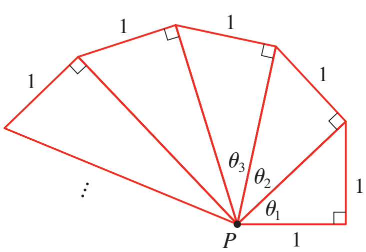



FIGURE FOR PROBLEM 7

**7.** Right-angled triangles are constructed as in the figure. Each triangle has height 1 and its base is the hypotenuse of the preceding triangle. Show that this sequence of triangles makes indefinitely many turns around $P$ by showing that $\sum \theta_n$ is a divergent series.

## Chapter 10

A Click here for answers.

S Click here for solutions.

**1.** A projectile of mass $m$ is fired from the origin at an angle of elevation $\alpha$. In addition to gravity, assume that air resistance provides a force that is proportional to the velocity and that opposes the motion. Then, by Newton's Second Law, the total force acting on the projectile satisfies the equation

$$\boxed{2} \qquad m\frac{d^2\mathbf{R}}{dt^2} = -mg\,\mathbf{j} - k\frac{d\mathbf{R}}{dt}$$

where $\mathbf{R}$ is the position vector and $k > 0$ is the constant of proportionality.

(a) Show that Equation 2 can be integrated to obtain the equation

$$\frac{d\mathbf{R}}{dt} + \frac{k}{m}\mathbf{R} = \mathbf{v}_0 - gt\,\mathbf{j}$$

where $\mathbf{v}_0 = \mathbf{v}(0) = \dfrac{d\mathbf{R}}{dt}(0)$.

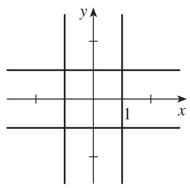
(b) Multiply both sides of the equation in part (a) by $e^{(k/m)t}$ and show that the left-hand side of the resulting equation is the derivative of the product $e^{(k/m)t}\,\mathbf{R}(t)$. Then integrate to find an expression for the position vector $\mathbf{R}(t)$.

# Answers

## Chapter 1

S Solutions

**1.**




## Chapter 3

S Solutions

**1.** (a) $[-1, 2]$ (b) $-1/\left(8\sqrt{3-x}\sqrt{2-\sqrt{3-x}}\sqrt{1-\sqrt{2-\sqrt{3-x}}}\right)$

**5.** $(1, -2), (-1, 0)$

## Chapter 4

S Solutions

**1.** $\frac{4}{3}$

**3.** (a) $-\tan\theta\left[\dfrac{1}{c}\dfrac{dc}{dt}+\dfrac{1}{b}\dfrac{db}{dt}\right]$ (b) $\dfrac{b\dfrac{db}{dt}+c\dfrac{dc}{dt}-\left(b\dfrac{dc}{dt}+c\dfrac{db}{dt}\right)\sec\theta}{\sqrt{b^2+c^2-2bc\cos\theta}}$

**5.** $(m/2, m^2/4)$

## Chapter 5

S Solutions

**3.** (a) $(n-1)n/2$ (b) $\frac{1}{2}[\![b]\!](2b-[\![b]\!]-1)-\frac{1}{2}[\![a]\!](2a-[\![a]\!]-1)$

**7.** $(b^b a^{-a})^{1/(b-a)}e^{-1}$

**9.** $2(\sqrt{2}-1)$

## Chapter 6

S Solutions

**1.** $2\pi/3 - \sqrt{3}/2$

**3.** (a) $V = \int_0^h \pi[f(y)]^2\,dy$

(c) $f(y) = \sqrt{kA/(\pi C)}\,y^{1/4}$

Advantage: the markings on the container are equally spaced.

**5.** (a) $P(z) = P_0 + g\int_0^z \rho(x)\,dx$ (b) $(P_0 - \rho_0 gH)(\pi r^2) + \rho_0 gHe^{L/H}\int_{-r}^{r} e^{x/H}\cdot 2\sqrt{r^2-x^2}\,dx$

**7.** $2/\pi, 1/\pi$

**9.** $y = \frac{32}{9}x^2$

**11.** $B = 16A$

## Chapter 8

S Solutions

**1.** (b) $0$ if $x = 0$, $(1/x) - \cot x$ if $x \neq k\pi$, $k$ an integer

**5.** $(-1, 1)$, $(x^3 + 4x^2 + x)/(1-x)^4$

## Chapter 10

S Solutions

**1.** (b) $\mathbf{R}(t) = (m/k)(1 - e^{-kt/m})\mathbf{v}_0 + (gm/k)[(m/k)(1 - e^{-kt/m}) - t]\mathbf{j}$

# Solutions

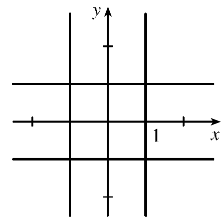
**E Exercises**

## Chapter 1

**1.** $|x| + |y| = 1 + |xy| \quad \Leftrightarrow \quad |xy| - |x| - |y| + 1 = 0 \quad \Leftrightarrow$

$|x|\,|y| - |x| - |y| + 1 = 0 \quad \Leftrightarrow \quad (|x| - 1)(|y| - 1) = 0 \quad \Leftrightarrow$

$x = \pm 1$ or $y = \pm 1$.




**E Exercises**

## Chapter 3

**1.** (a) $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}$

$$D = \left\{x \mid 3 - x \geq 0, 2 - \sqrt{3 - x} \geq 0, 1 - \sqrt{2 - \sqrt{3 - x}} \geq 0\right\} = \left\{x \mid 3 \geq x, 2 \geq \sqrt{3 - x}, 1 \geq \sqrt{2 - \sqrt{3 - x}}\right\}$$

$$= \{x \mid 3 \geq x,\ 4 \geq 3 - x, 1 \geq 2 - \sqrt{3 - x}\} = \{x \mid x \leq 3, x \geq -1, 1 \leq \sqrt{3 - x}\}$$

$$= \{x \mid x \leq 3, x \geq -1, 1 \leq 3 - x\} = \{x \mid x \leq 3, x \geq -1, x \leq 2\} = \{x \mid -1 \leq x \leq 2\} = [-1, 2]$$

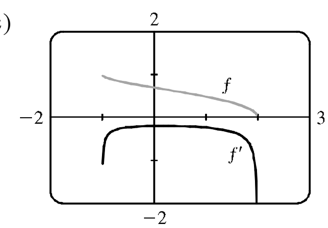
(b) $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - x}}} \quad \Rightarrow$

$$f'(x) = \frac{1}{2\sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}} \frac{d}{dx}\left(1 - \sqrt{2 - \sqrt{3 - x}}\right)$$

$$= \frac{1}{2\sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}} \cdot \frac{-1}{2\sqrt{2 - \sqrt{3 - x}}} \frac{d}{dx}\left(2 - \sqrt{3 - x}\right)$$

$$= -\frac{1}{8\sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}\,\sqrt{2 - \sqrt{3 - x}}\,\sqrt{3 - x}}$$

(c)




Note that $f$ is always decreasing and $f'$ is always negative.

**3.** We use mathematical induction. Let $S_n$ be the statement that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

$S_1$ is true because

$$\frac{d}{dx}(\sin^4 x + \cos^4 x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x = 4\sin x\cos x\,(\sin^2 x - \cos^2 x)$$

$$= -4\sin x\cos x\cos 2x = -2\sin 2x\cos 2x = -\sin 4x = \sin(-4x)$$

$$= \cos\left(\tfrac{\pi}{2} - (-4x)\right) = \cos\left(\tfrac{\pi}{2} + 4x\right) = 4^{n-1}\cos\left(4x + n\tfrac{\pi}{2}\right) \quad \text{when } n = 1$$

Now assume $S_k$ is true, that is, $\dfrac{d^k}{dx^k}(\sin^4 x + \cos^4 x) = 4^{k-1}\cos\left(4x + k\frac{\pi}{2}\right)$. Then

$$\frac{d^{k+1}}{dx^{k+1}}(\sin^4 x + \cos^4 x) = \frac{d}{dx}\left[\frac{d^k}{dx^k}(\sin^4 x + \cos^4 x)\right] = \frac{d}{dx}\left[4^{k-1}\cos\left(4x + k\tfrac{\pi}{2}\right)\right]$$

$$= -4^{k-1}\sin\left(4x + k\tfrac{\pi}{2}\right) \cdot \tfrac{d}{dx}\left(4x + k\tfrac{\pi}{2}\right) = -4^k\sin\left(4x + k\tfrac{\pi}{2}\right)$$

$$= 4^k\sin\left(-4x - k\tfrac{\pi}{2}\right) = 4^k\cos\left(\tfrac{\pi}{2} - \left(-4x - k\tfrac{\pi}{2}\right)\right) = 4^k\cos\left(4x + (k+1)\tfrac{\pi}{2}\right)$$

which shows that $S_{k+1}$ is true.

Therefore, $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right)$ for every positive integer $n$, by mathematical induction.

*Another proof:* First write

$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = 1 - \frac{1}{2}\sin^2 2x = 1 - \frac{1}{4}(1 - \cos 4x) = \frac{3}{4} + \frac{1}{4}\cos 4x.$

Then we have $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = \dfrac{d^n}{dx^n}\left(\frac{3}{4} + \frac{1}{4}\cos 4x\right) = \frac{1}{4}\cdot 4^n\cos\left(4x + n\frac{\pi}{2}\right) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right).$

**5.** $y = x^4 - 2x^2 - x \quad\Rightarrow\quad y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is $y - (a^4 - 2a^2 - a) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + (-3a^4 + 2a^2)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \quad\Rightarrow\quad (a - b)(a^2 + ab + b^2) = (a - b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \quad\Rightarrow\quad 3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives $1 = a^2 - a^2 + a^2 \quad\Rightarrow\quad a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ gives $3(a^2 + b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$. Thus, $ab = (a^2 + ab + b^2) - (a^2 + b^2) = 1 - \frac{2}{3} = \frac{1}{3}$, so $b = \dfrac{1}{3a}$. Hence, $a^2 + \dfrac{1}{9a^2} = \dfrac{2}{3}$, so $9a^4 + 1 = 6a^2 \quad\Rightarrow$ $0 = 9a^4 - 6a^2 + 1 = (3a^2 - 1)^2$. So $3a^2 - 1 = 0 \quad\Rightarrow\quad a^2 = \frac{1}{3} \quad\Rightarrow\quad b^2 = \dfrac{1}{9a^2} = \frac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

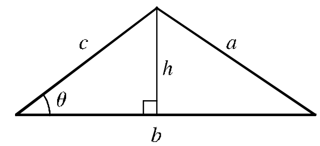
## E Exercises Chapter 4

**1.** $f(x) = \dfrac{1}{1 + |x|} + \dfrac{1}{1 + |x - 2|}$

$$= \begin{cases} \dfrac{1}{1 - x} + \dfrac{1}{1 - (x - 2)} & \text{if } x < 0 \\[2mm] \dfrac{1}{1 + x} + \dfrac{1}{1 - (x - 2)} & \text{if } 0 \leq x < 2 \\[2mm] \dfrac{1}{1 + x} + \dfrac{1}{1 + (x - 2)} & \text{if } x \geq 2 \end{cases} \quad\Rightarrow\quad f'(x) = \begin{cases} \dfrac{1}{(1 - x)^2} + \dfrac{1}{(3 - x)^2} & \text{if } x < 0 \\[2mm] \dfrac{-1}{(1 + x)^2} + \dfrac{1}{(3 - x)^2} & \text{if } 0 < x < 2 \\[2mm] \dfrac{-1}{(1 + x)^2} - \dfrac{1}{(x - 1)^2} & \text{if } x > 2 \end{cases}$$

We see that $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 2$. For $0 < x < 2$, we have

$f'(x) = \dfrac{1}{(3 - x)^2} - \dfrac{1}{(x + 1)^2} = \dfrac{(x^2 + 2x + 1) - (x^2 - 6x + 9)}{(3 - x)^2(x + 1)^2} = \dfrac{8(x - 1)}{(3 - x)^2(x + 1)^2}$, so $f'(x) < 0$ for $0 < x < 1$, $f'(1) = 0$ and $f'(x) > 0$ for $1 < x < 2$. We have shown that $f'(x) > 0$ for $x < 0$; $f'(x) < 0$ for $0 < x < 1$; $f'(x) > 0$ for $1 < x < 2$; and $f'(x) < 0$ for $x > 2$. Therefore, by the First Derivative Test, the local maxima of $f$ are at $x = 0$ and $x = 2$, where $f$ takes the value $\frac{4}{3}$. Therefore, $\frac{4}{3}$ is the absolute maximum value of $f$.




**3.** (a) $A = \frac{1}{2}bh$ with $\sin\theta = h/c$, so $A = \frac{1}{2}bc\sin\theta$. But $A$ is a constant, so differentiating this equation with respect to $t$, we get

$$\frac{dA}{dt} = 0 = \frac{1}{2}\left[bc\cos\theta\,\frac{d\theta}{dt} + b\,\frac{dc}{dt}\sin\theta + \frac{db}{dt}\,c\sin\theta\right] \quad\Rightarrow$$

$$bc\cos\theta\,\frac{d\theta}{dt} = -\sin\theta\left[b\,\frac{dc}{dt} + c\,\frac{db}{dt}\right] \quad\Rightarrow\quad \frac{d\theta}{dt} = -\tan\theta\left[\frac{1}{c}\frac{dc}{dt} + \frac{1}{b}\frac{db}{dt}\right].$$

(b) We use the Law of Cosines to get the length of side $a$ in terms of those of $b$ and $c$, and then we differentiate implicitly with respect to $t$: $a^2 = b^2 + c^2 - 2bc\cos\theta \quad\Rightarrow\quad 2a\,\frac{da}{dt} = 2b\,\frac{db}{dt} + 2c\,\frac{dc}{dt} - 2\left[bc(-\sin\theta)\,\frac{d\theta}{dt} + b\,\frac{dc}{dt}\cos\theta + \frac{db}{dt}\,c\cos\theta\right]$

$\Rightarrow\quad \dfrac{da}{dt} = \dfrac{1}{a}\left(b\,\dfrac{db}{dt} + c\,\dfrac{dc}{dt} + bc\sin\theta\,\dfrac{d\theta}{dt} - b\,\dfrac{dc}{dt}\cos\theta - c\,\dfrac{db}{dt}\cos\theta\right)$. Now we substitute our value of $a$ from the Law of Cosines and the value of $d\theta/dt$ from part (a), and simplify (primes signify differentiation by $t$):

$$\frac{da}{dt} = \frac{bb' + cc' + bc\sin\theta\,[-\tan\theta(c'/c + b'/b)] - (bc' + cb')(\cos\theta)}{\sqrt{b^2 + c^2 - 2bc\,\cos\theta}}$$

$$= \frac{bb' + cc' - [\sin^2\theta(bc' + cb') + \cos^2\theta(bc' + cb')]/\cos\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}} = \frac{bb' + cc' - (bc' + cb')\sec\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}}$$

**5.** $A = (x_1, x_1^2)$ and $B = (x_2, x_2^2)$, where $x_1$ and $x_2$ are the solutions of the quadratic equation $x^2 = mx + b$. Let $P = (x, x^2)$ and set $A_1 = (x_1, 0)$, $B_1 = (x_2, 0)$, and $P_1 = (x, 0)$. Let $f(x)$ denote the area of triangle $PAB$. Then $f(x)$ can be expressed in terms of the areas of three trapezoids as follows:

$$\begin{aligned} f(x) &= \text{area}\,(A_1ABB_1) - \text{area}\,(A_1APP_1) - \text{area}\,(B_1BPP_1) \\ &= \tfrac{1}{2}\left(x_1^2 + x_2^2\right)(x_2 - x_1) - \tfrac{1}{2}\left(x_1^2 + x^2\right)(x - x_1) - \tfrac{1}{2}\left(x^2 + x_2^2\right)(x_2 - x) \end{aligned}$$

After expanding and canceling terms, we get

$f(x) = \frac{1}{2}\left(x_2x_1^2 - x_1x_2^2 - xx_1^2 + x_1x^2 - x_2x^2 + xx_2^2\right) = \frac{1}{2}\left[x_1^2(x_2 - x) + x_2^2(x - x_1) + x^2(x_1 - x_2)\right]$

$f'(x) = \frac{1}{2}\left[-x_1^2 + x_2^2 + 2x(x_1 - x_2)\right]$. $\quad f''(x) = \frac{1}{2}[2(x_1 - x_2)] = x_1 - x_2 < 0$ since $x_2 > x_1$.

$f'(x) = 0 \quad\Rightarrow\quad 2x(x_1 - x_2) = x_1^2 - x_2^2 \quad\Rightarrow\quad x_P = \frac{1}{2}(x_1 + x_2)$.

$$\begin{aligned} f(x_P) &= \tfrac{1}{2}\left(x_1^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + x_2^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + \tfrac{1}{4}(x_1 + x_2)^2(x_1 - x_2)\right) \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}(x_2 - x_1)\left(x_1^2 + x_2^2\right) - \tfrac{1}{4}(x_2 - x_1)(x_1 + x_2)^2\right] = \tfrac{1}{8}(x_2 - x_1)\left[2\left(x_1^2 + x_2^2\right) - \left(x_1^2 + 2x_1x_2 + x_2^2\right)\right] \\ &= \tfrac{1}{8}(x_2 - x_1)\left(x_1^2 - 2x_1x_2 + x_2^2\right) = \tfrac{1}{8}(x_2 - x_1)(x_1 - x_2)^2 = \tfrac{1}{8}(x_2 - x_1)(x_2 - x_1)^2 \\ &= \tfrac{1}{8}(x_2 - x_1)^3 \end{aligned}$$

To put this in terms of $m$ and $b$, we solve the system $y = x_1^2$ and $y = mx_1 + b$, giving us $x_1^2 - mx_1 - b = 0 \quad\Rightarrow$ $x_1 = \frac{1}{2}\left(m - \sqrt{m^2 + 4b}\right)$. Similarly, $x_2 = \frac{1}{2}\left(m + \sqrt{m^2 + 4b}\right)$. The area is then $\frac{1}{8}(x_2 - x_1)^3 = \frac{1}{8}\left(\sqrt{m^2 + 4b}\right)^3$, and is attained at the point $P\left(x_P, x_P^2\right) = P\left(\frac{1}{2}m, \frac{1}{4}m^2\right)$.

*Note:* Another way to get an expression for $f(x)$ is to use the formula for an area of a triangle in terms of the coordinates of the vertices: $f(x) = \frac{1}{2}\left[\left(x_2x_1^2 - x_1x_2^2\right) + \left(x_1x^2 - xx_1^2\right) + \left(xx_2^2 - x_2x^2\right)\right]$.

## E Exercises Chapter 5

**1.** For $1 \le x \le 2$, we have $x^4 \le 2^4 = 16$, so $1 + x^4 \le 17$ and $\dfrac{1}{1+x^4} \ge \dfrac{1}{17}$. Thus, $\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx \ge \int_1^2 \frac{1}{17}\,dx = \frac{1}{17}$.

Also $1 + x^4 > x^4$ for $1 \le x \le 2$, so $\dfrac{1}{1+x^4} < \dfrac{1}{x^4}$ and $\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx < \int_1^2 x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_1^2 = -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}$.

Thus, we have the estimate $\displaystyle\frac{1}{17} \le \int_1^2 \frac{1}{1+x^4}\,dx \le \frac{7}{24}$.

**3.** (a) We can split the integral $\int_0^n [\![x]\!]\,dx$ into the sum $\sum\limits_{i=1}^{n} \left[\int_{i-1}^{i} [\![x]\!]\,dx\right]$. But on each of the intervals $[i-1, i)$ of integration, $[\![x]\!]$ is a constant function, namely $i-1$. So the $i$th integral in the sum is equal to $(i-1)[i-(i-1)] = (i-1)$. So the original integral is equal to $\sum\limits_{i=1}^{n}(i-1) = \sum\limits_{i=1}^{n-1} i = \dfrac{(n-1)n}{2}$.

(b) We can write $\int_a^b [\![x]\!]\,dx = \int_0^b [\![x]\!]\,dx - \int_0^a [\![x]\!]\,dx$. Now $\int_0^b [\![x]\!]\,dx = \int_0^{[\![b]\!]} [\![x]\!]\,dx + \int_{[\![b]\!]}^b [\![x]\!]\,dx$. The first of these integrals is equal to $\frac{1}{2}([\![b]\!]-1)\,[\![b]\!]$, by part (a), and since $[\![x]\!] = [\![b]\!]$ on $[[\![b]\!], b]$, the second integral is just $[\![b]\!]\,(b - [\![b]\!])$. So $\int_0^b [\![x]\!]\,dx = \frac{1}{2}([\![b]\!]-1)\,[\![b]\!] + [\![b]\!]\,(b-[\![b]\!]) = \frac{1}{2}[\![b]\!]\,(2b-[\![b]\!]-1)$ and similarly $\int_0^a [\![x]\!]\,dx = \frac{1}{2}[\![a]\!]\,(2a-[\![a]\!]-1)$.

Therefore, $\int_a^b [\![x]\!]\,dx = \frac{1}{2}[\![b]\!]\,(2b-[\![b]\!]-1) - \frac{1}{2}[\![a]\!]\,(2a-[\![a]\!]-1)$.

**5.** Note that $\displaystyle\frac{d}{dx}\left(\int_0^x \left[\int_0^u f(t)\,dt\right] du\right) = \int_0^x f(t)\,dt$ by FTC1, while

$$\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] = \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\left[\int_0^x f(u)u\,du\right] = \textstyle\int_0^x f(u)\,du + xf(x) - f(x)x = \int_0^x f(u)\,du$$

Hence, $\int_0^x f(u)(x-u)\,du = \int_0^x \left[\int_0^u f(t)\,dt\right] du + C$. Setting $x = 0$ gives $C = 0$.

**7.** $0 < a < b$. Now

$$\int_0^1 [bx + a(1-x)]^t\,dx = \int_a^b \frac{u^t}{(b-a)}\,du \quad [\text{put } u = bx + a(1-x)] \quad = \left[\frac{u^{t+1}}{(t+1)(b-a)}\right]_a^b = \frac{b^{t+1}-a^{t+1}}{(t+1)(b-a)}.$$

Now let $y = \displaystyle\lim_{t\to 0}\left[\frac{b^{t+1}-a^{t+1}}{(t+1)(b-a)}\right]^{1/t}$. Then $\ln y = \displaystyle\lim_{t\to 0}\left[\frac{1}{t}\ln\frac{b^{t+1}-a^{t+1}}{(t+1)(b-a)}\right]$.

This limit is of the form $0/0$, so we can apply l'Hospital's Rule to get

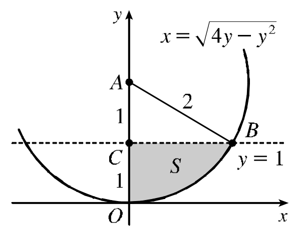
$$\ln y = \lim_{t\to 0}\left[\frac{b^{t+1}\ln b - a^{t+1}\ln a}{b^{t+1}-a^{t+1}} - \frac{1}{t+1}\right] = \frac{b\ln b - a\ln a}{b-a} - 1 = \frac{b\ln b}{b-a} - \frac{a\ln a}{b-a} - \ln e = \ln\frac{b^{b/(b-a)}}{ea^{a/(b-a)}}.$$

Therefore, $y = e^{-1}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$.

**9.** $$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{n}{n+1}}+\sqrt{\frac{n}{n+2}}+\cdots+\sqrt{\frac{n}{n+n}}\right)$$
$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}}+\frac{1}{\sqrt{1+2/n}}+\cdots+\frac{1}{\sqrt{1+1}}\right)$$
$$=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{i}{n}\right)\qquad\left[\text{where } f(x)=\frac{1}{\sqrt{1+x}}\right]$$
$$=\int_0^1\frac{1}{\sqrt{1+x}}\,dx=\left[2\sqrt{1+x}\right]_0^1=2\left(\sqrt{2}-1\right)$$

## E Exercises Chapter 6

**1.** $x^2+y^2\le 4y \quad\Leftrightarrow\quad x^2+(y-2)^2\le 4$, so $S$ is part of a circle, as shown in the diagram. The area of $S$ is




$$\int_0^1\sqrt{4y-y^2}\,dy\overset{113}{=}\left[\tfrac{y-2}{2}\sqrt{4y-y^2}+2\cos^{-1}\left(\tfrac{2-y}{2}\right)\right]_0^1\qquad[a=2]$$
$$=-\tfrac{1}{2}\sqrt{3}+2\cos^{-1}\left(\tfrac{1}{2}\right)-2\cos^{-1}1$$
$$=-\tfrac{\sqrt{3}}{2}+2\left(\tfrac{\pi}{3}\right)-2(0)=\tfrac{2\pi}{3}-\tfrac{\sqrt{3}}{2}$$

*Another method (without calculus):* Note that $\theta=\angle CAB=\frac{\pi}{3}$, so the area is

$$(\text{area of sector } OAB)-(\text{area of }\triangle ABC)=\tfrac{1}{2}(2^2)\tfrac{\pi}{3}-\tfrac{1}{2}(1)\sqrt{3}=\tfrac{2\pi}{3}-\tfrac{\sqrt{3}}{2}$$

**3.** (a) Stacking disks along the $y$-axis gives us $V=\int_0^h\pi\,[f(y)]^2\,dy$.

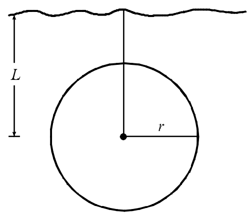
(b) Using the Chain Rule, $\dfrac{dV}{dt}=\dfrac{dV}{dh}\cdot\dfrac{dh}{dt}=\pi\,[f(h)]^2\,\dfrac{dh}{dt}$.

(c) $kA\sqrt{h}=\pi[f(h)]^2\,\dfrac{dh}{dt}$. Set $\dfrac{dh}{dt}=C$: $\pi[f(h)]^2\,C=kA\sqrt{h}\quad\Rightarrow\quad[f(h)]^2=\dfrac{kA}{\pi C}\sqrt{h}\quad\Rightarrow\quad f(h)=\sqrt{\dfrac{kA}{\pi C}}\,h^{1/4}$;

that is, $f(y)=\sqrt{\dfrac{kA}{\pi C}}\,y^{1/4}$. The advantage of having $\dfrac{dh}{dt}=C$ is that the markings on the container are equally spaced.

**5.** (a) Choose a vertical $x$-axis pointing downward with its origin at the surface. In order to calculate the pressure at depth $z$, consider $n$ subintervals of the interval $[0,z]$ by points $x_i$ and choose a point $x_i^*\in[x_{i-1},x_i]$ for each $i$. The thin layer of water lying between depth $x_{i-1}$ and depth $x_i$ has a density of approximately $\rho(x_i^*)$, so the weight of a piece of that layer with unit cross-sectional area is $\rho(x_i^*)g\,\Delta x$. The total weight of a column of water extending from the surface to depth $z$ (with unit cross-sectional area) would be approximately $\sum_{i=1}^{n}\rho(x_i^*)g\,\Delta x$. The estimate becomes exact if we take the limit as $n\to\infty$; weight (or force) per unit area at depth $z$ is $W=\lim_{n\to\infty}\sum_{i=1}^{n}\rho(x_i^*)g\,\Delta x$. In other words, $P(z)=\int_0^z\rho(x)g\,dx$. More generally, if we make no assumptions about the location of the origin, then $P(z)=P_0+\int_0^z\rho(x)g\,dx$, where $P_0$ is the pressure at $x=0$. Differentiating, we get $dP/dz=\rho(z)g$.

(b)




$$
\begin{aligned}
F &= \int_{-r}^{r} P(L+x)\cdot 2\sqrt{r^2-x^2}\,dx \\
&= \int_{-r}^{r}\left(P_0+\int_0^{L+x}\rho_0 e^{z/H} g\,dz\right)\cdot 2\sqrt{r^2-x^2}\,dx \\
&= P_0\int_{-r}^{r} 2\sqrt{r^2-x^2}\,dx+\rho_0 gH\int_{-r}^{r}\left(e^{(L+x)/H}-1\right)\cdot 2\sqrt{r^2-x^2}\,dx \\
&= (P_0-\rho_0 gH)\int_{-r}^{r} 2\sqrt{r^2-x^2}\,dx+\rho_0 gH\int_{-r}^{r} e^{(L+x)/H}\cdot 2\sqrt{r^2-x^2}\,dx \\
&= (P_0-\rho_0 gH)(\pi r^2)+\rho_0 gHe^{L/H}\int_{-r}^{r} e^{x/H}\cdot 2\sqrt{r^2-x^2}\,dx
\end{aligned}
$$

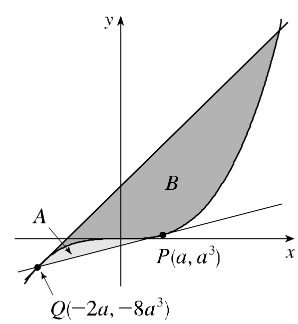
**7.** If $h=L$, then $P=\dfrac{\text{area under } y=L\sin\theta}{\text{area of rectangle}}=\dfrac{\int_0^\pi L\sin\theta\,d\theta}{\pi L}=\dfrac{[-\cos\theta]_0^\pi}{\pi}=\dfrac{-(-1)+1}{\pi}=\dfrac{2}{\pi}$

If $h=L/2$, then $P=\dfrac{\text{area under } y=\frac{1}{2}L\sin\theta}{\text{area of rectangle}}=\dfrac{\int_0^\pi \frac{1}{2}L\sin\theta\,d\theta}{\pi L}=\dfrac{[-\cos\theta]_0^\pi}{2\pi}=\dfrac{2}{2\pi}=\dfrac{1}{\pi}$.

**9.** We must find expressions for the areas $A$ and $B$, and then set them equal and see what this says about the curve $C$. If $P=(a,2a^2)$, then area $A$ is just $\int_0^a (2x^2-x^2)\,dx=\int_0^a x^2\,dx=\frac{1}{3}a^3$. To find area $B$, we use $y$ as the variable of integration. So we find the equation of the middle curve as a function of $y$: $y=2x^2 \Leftrightarrow x=\sqrt{y/2}$, since we are concerned with the first quadrant only. We can express area $B$ as

$\int_0^{2a^2}\left[\sqrt{y/2}-C(y)\right]dy=\left[\frac{4}{3}(y/2)^{3/2}\right]_0^{2a^2}-\int_0^{2a^2}C(y)\,dy=\frac{4}{3}a^3-\int_0^{2a^2}C(y)\,dy$, where $C(y)$ is the function with graph $C$. Setting $A=B$, we get $\frac{1}{3}a^3=\frac{4}{3}a^3-\int_0^{2a^2}C(y)\,dy \Leftrightarrow \int_0^{2a^2}C(y)\,dy=a^3$. Now we differentiate this equation with respect to $a$ using the Chain Rule and the Fundamental Theorem: $C(2a^2)(4a)=3a^2 \Rightarrow C(y)=\frac{3}{4}\sqrt{y/2}$, where $y=2a^2$. Now we can solve for $y$: $x=\frac{3}{4}\sqrt{y/2} \Rightarrow x^2=\frac{9}{16}(y/2) \Rightarrow y=\frac{32}{9}x^2$.

**11.** We assume that $P$ lies in the region of positive $x$. Since $y=x^3$ is an odd function, this assumption will not affect the result of the calculation. Let $P=(a,a^3)$. The slope of the tangent to the curve $y=x^3$ at $P$ is $3a^2$, and so the equation of the tangent is $y-a^3=3a^2(x-a) \Leftrightarrow y=3a^2x-2a^3$. We solve this simultaneously with $y=x^3$ to find the other point of intersection: $x^3=3a^2x-2a^3 \Leftrightarrow (x-a)^2(x+2a)=0$. So $Q=(-2a,-8a^3)$ is the other point of intersection. The equation of the tangent at $Q$ is




$y-(-8a^3)=12a^2[x-(-2a)] \Leftrightarrow y=12a^2x+16a^3$. By symmetry, this tangent will intersect the curve again at $x=-2(-2a)=4a$. The curve lies above the first tangent, and below the second, so we are looking for a relationship between $A=\int_{-2a}^{a}\left[x^3-(3a^2x-2a^3)\right]dx$ and $B=\int_{-2a}^{4a}\left[(12a^2x+16a^3)-x^3\right]dx$. We calculate $A=\left[\frac{1}{4}x^4-\frac{3}{2}a^2x^2+2a^3x\right]_{-2a}^{a}=\frac{3}{4}a^4-(-6a^4)=\frac{27}{4}a^4$, and $B=\left[6a^2x^2+16a^3x-\frac{1}{4}x^4\right]_{-2a}^{4a}=96a^4-(-12a^4)=108a^4$. We see that $B=16A=2^4A$. This is because our calculation of area $B$ was essentially the same as that of area $A$, with $a$ replaced by $-2a$, so if we replace $a$ with $-2a$ in our expression for $A$, we get $\frac{27}{4}(-2a)^4=108a^4=B$.

**E Exercises** **Chapter 8**

**1.** (a) From Formula 14a in Appendix C, with $x = y = \theta$, we get $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$, so $\cot 2\theta = \dfrac{1-\tan^2\theta}{2\tan\theta} \Rightarrow$

$2\cot 2\theta = \dfrac{1-\tan^2\theta}{\tan\theta} = \cot\theta - \tan\theta$. Replacing $\theta$ by $\frac{1}{2}x$, we get $2\cot x = \cot\frac{1}{2}x - \tan\frac{1}{2}x$, or

$\tan\frac{1}{2}x = \cot\frac{1}{2}x - 2\cot x$.

(b) From part (a) with $\dfrac{x}{2^{n-1}}$ in place of $x$, $\tan\dfrac{x}{2^n} = \cot\dfrac{x}{2^n} - 2\cot\dfrac{x}{2^{n-1}}$, so the $n$th partial sum of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}\tan\frac{x}{2^n}$ is

$$\begin{aligned} s_n &= \frac{\tan(x/2)}{2} + \frac{\tan(x/4)}{4} + \frac{\tan(x/8)}{8} + \cdots + \frac{\tan(x/2^n)}{2^n} \\ &= \left[\frac{\cot(x/2)}{2} - \cot x\right] + \left[\frac{\cot(x/4)}{4} - \frac{\cot(x/2)}{2}\right] + \left[\frac{\cot(x/8)}{8} - \frac{\cot(x/4)}{4}\right] + \cdots \\ &\qquad + \left[\frac{\cot(x/2^n)}{2^n} - \frac{\cot(x/2^{n-1})}{2^{n-1}}\right] = -\cot x + \frac{\cot(x/2^n)}{2^n} \qquad \text{[telescoping sum]} \end{aligned}$$

Now $\dfrac{\cot(x/2^n)}{2^n} = \dfrac{\cos(x/2^n)}{2^n\sin(x/2^n)} = \dfrac{\cos(x/2^n)}{x}\cdot\dfrac{x/2^n}{\sin(x/2^n)} \to \dfrac{1}{x}\cdot 1 = \dfrac{1}{x}$ as $n\to\infty$ since $x/2^n \to 0$ for $x \neq 0$.

Therefore, if $x \neq 0$ and $x \neq k\pi$ where $k$ is any integer, then

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\tan\frac{x}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(-\cot x + \frac{1}{2^n}\cot\frac{x}{2^n}\right) = -\cot x + \frac{1}{x}$$

If $x = 0$, then all terms in the series are 0, so the sum is 0.

**3.** (a) Let $a = \arctan x$ and $b = \arctan y$. Then, from Formula 14b in Appendix C,

$$\tan(a-b) = \frac{\tan a - \tan b}{1+\tan a\tan b} = \frac{\tan(\arctan x) - \tan(\arctan y)}{1+\tan(\arctan x)\tan(\arctan y)} = \frac{x-y}{1+xy}$$

Now $\arctan x - \arctan y = a - b = \arctan(\tan(a-b)) = \arctan\dfrac{x-y}{1+xy}$ since $-\frac{\pi}{2} < a - b < \frac{\pi}{2}$.

(b) From part (a) we have

$$\arctan\tfrac{120}{119} - \arctan\tfrac{1}{239} = \arctan\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}} = \arctan\frac{\frac{28{,}561}{28{,}441}}{\frac{28{,}561}{28{,}441}} = \arctan 1 = \tfrac{\pi}{4}$$

(c) Replacing $y$ by $-y$ in the formula of part (a), we get $\arctan x + \arctan y = \arctan\dfrac{x+y}{1-xy}$. So

$$\begin{aligned} 4\arctan\tfrac{1}{5} &= 2\left(\arctan\tfrac{1}{5} + \arctan\tfrac{1}{5}\right) = 2\arctan\frac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}\cdot\frac{1}{5}} = 2\arctan\tfrac{5}{12} = \arctan\tfrac{5}{12} + \arctan\tfrac{5}{12} \\ &= \arctan\frac{\frac{5}{12}+\frac{5}{12}}{1-\frac{5}{12}\cdot\frac{5}{12}} = \arctan\tfrac{120}{119} \end{aligned}$$

Thus, from part (b), we have $4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \arctan\frac{120}{119} - \arctan\frac{1}{239} = \frac{\pi}{4}$.

(d) From Example 7 in Section 8.6 we have $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \dfrac{x^{11}}{11} + \cdots$, so

$$\arctan\frac{1}{5} = \frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \frac{1}{9\cdot 5^9} - \frac{1}{11\cdot 5^{11}} + \cdots$$

This is an alternating series and the size of the terms decreases to 0, so by the Alternating Series Estimation Theorem, the sum lies between $s_5$ and $s_6$, that is, $0.197395560 < \arctan\frac{1}{5} < 0.197395562$.

(e) From the series in part (d) we get $\arctan \frac{1}{239} = \frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \cdots$. The third term is less than $2.6 \times 10^{-13}$, so by the Alternating Series Estimation Theorem, we have, to nine decimal places, $\arctan \frac{1}{239} \approx s_2 \approx 0.004184076$. Thus, $0.004184075 < \arctan \frac{1}{239} < 0.004184077$.

(f) From part (c) we have $\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$, so from parts (d) and (e) we have $16(0.197395560) - 4(0.004184077) < \pi < 16(0.197395562) - 4(0.004184075) \Rightarrow$ $3.141592652 < \pi < 3.141592692$. So, to 7 decimal places, $\pi \approx 3.1415927$.

**5.** We start with the geometric series $\sum\limits_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$, $|x| < 1$, and differentiate:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} \text{ for } |x| < 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}$$

for $|x| < 1$. Differentiate again:

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{(1-x)^2 - x \cdot 2(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^2 x^n = \frac{x^2+x}{(1-x)^3} \quad \Rightarrow$$

$$\sum_{n=1}^{\infty} n^3 x^{n-1} = \frac{d}{dx}\frac{x^2+x}{(1-x)^3} = \frac{(1-x)^3(2x+1) - \left(x^2+x\right)3(1-x)^2(-1)}{(1-x)^6} = \frac{x^2+4x+1}{(1-x)^4} \quad \Rightarrow$$

$\sum\limits_{n=1}^{\infty} n^3 x^n = \dfrac{x^3+4x^2+x}{(1-x)^4}$, $|x| < 1$. The radius of convergence is 1 because that is the radius of convergence for the geometric series we started with. If $x = \pm 1$, the series is $\sum n^3 (\pm 1)^n$, which diverges by the Test For Divergence, so the interval of convergence is $(-1, 1)$.

**7.** Suppose the base of the first right triangle has length $a$. Then by repeated use of the Pythagorean theorem, we find that the base of the second right triangle has length $\sqrt{1+a^2}$, the base of the third right triangle has length $\sqrt{2+a^2}$, and in general, the $n$th right triangle has base of length $\sqrt{n-1+a^2}$ and hypotenuse of length $\sqrt{n+a^2}$. Thus, $\theta_n = \tan^{-1}\left(1/\sqrt{n-1+a^2}\right)$ and $\sum\limits_{n=1}^{\infty} \theta_n = \sum\limits_{n=1}^{\infty} \tan^{-1}\left(\dfrac{1}{\sqrt{n-1+a^2}}\right) = \sum\limits_{n=0}^{\infty} \tan^{-1}\left(\dfrac{1}{\sqrt{n+a^2}}\right)$. We wish to show that this series diverges.

First notice that the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n+a^2}}$ diverges by the Limit Comparison Test with the divergent $p$-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ $\left(p = \frac{1}{2} \leq 1\right)$ since

$$\lim_{n\to\infty} \frac{1/\sqrt{n+a^2}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+a^2}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+a^2}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+a^2/n}} = 1 > 0. \text{ Thus, } \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+a^2}} \text{ also diverges.}$$

Now $\sum\limits_{n=0}^{\infty} \tan^{-1}\left(\dfrac{1}{\sqrt{n+a^2}}\right)$ diverges by the Limit Comparison Test with $\sum\limits_{n=0}^{\infty} \dfrac{1}{\sqrt{n+a^2}}$ since

$$\lim_{n\to\infty} \frac{\tan^{-1}\left(1/\sqrt{n+a^2}\right)}{1/\sqrt{n+a^2}} = \lim_{x\to\infty} \frac{\tan^{-1}\left(1/\sqrt{x+a^2}\right)}{1/\sqrt{x+a^2}} = \lim_{y\to\infty} \frac{\tan^{-1}(1/y)}{1/y} \qquad \left[y = \sqrt{x+a^2}\,\right]$$

$$= \lim_{z\to 0^+} \frac{\tan^{-1} z}{z} \quad [z = 1/y] \overset{\text{H}}{=} \lim_{z\to 0^+} \frac{1/(1+z^2)}{1} = 1 > 0$$

Thus, $\sum\limits_{n=1}^{\infty} \theta_n$ is a divergent series.

## E Exercises

### Chapter 10

1. (a) $m\dfrac{d^2\mathbf{R}}{dt^2} = -mg\,\mathbf{j} - k\dfrac{d\mathbf{R}}{dt} \quad\Rightarrow\quad \dfrac{d}{dt}\left(m\dfrac{d\mathbf{R}}{dt} + k\,\mathbf{R} + mgt\,\mathbf{j}\right) = \mathbf{0} \quad\Rightarrow\quad m\dfrac{d\mathbf{R}}{dt} + k\mathbf{R} + mgt\,\mathbf{j} = \mathbf{c}$ ($\mathbf{c}$ is a constant vector in the $xy$-plane). At $t = 0$, this says that $m\mathbf{v}\,(0) + k\mathbf{R}\,(0) = \mathbf{c}$. Since $\mathbf{v}\,(0) = \mathbf{v}_0$ and $\mathbf{R}\,(0) = \mathbf{0}$, we have $\mathbf{c} = m\mathbf{v}_0$. Therefore $\dfrac{d\mathbf{R}}{dt} + \dfrac{k}{m}\mathbf{R} + gt\,\mathbf{j} = \mathbf{v}_0$, or $\dfrac{d\mathbf{R}}{dt} + \dfrac{k}{m}\mathbf{R} = \mathbf{v}_0 - gt\,\mathbf{j}$.

   (b) Multiplying by $e^{(k/m)t}$ gives $e^{(k/m)t}\dfrac{d\mathbf{R}}{dt} + \dfrac{k}{m}e^{(k/m)t}\mathbf{R} = e^{(k/m)t}\mathbf{v}_0 - gte^{(k/m)t}\,\mathbf{j}$ or

   $\dfrac{d}{dt}\left(e^{(k/m)t}\mathbf{R}\right) = e^{(k/m)t}\mathbf{v}_0 - gte^{(k/m)t}\,\mathbf{j}$. Integrating gives

   $e^{(k/m)t}\mathbf{R} = \dfrac{m}{k}e^{(k/m)t}\mathbf{v}_0 - \left[\dfrac{mg}{k}te^{(k/m)t} - \dfrac{m^2g}{k^2}e^{(k/m)t}\right]\mathbf{j} + \mathbf{b}$ for some constant vector $\mathbf{b}$.

   Setting $t = 0$ yields the relation $\mathbf{R}\,(0) = \dfrac{m}{k}\mathbf{v}_0 + \dfrac{m^2g}{k^2}\,\mathbf{j} + \mathbf{b}$, so $\mathbf{b} = -\dfrac{m}{k}\mathbf{v}_0 - \dfrac{m^2g}{k^2}\,\mathbf{j}$.

   Thus $e^{(k/m)t}\mathbf{R} = \dfrac{m}{k}\left[e^{(k/m)t} - 1\right]\mathbf{v}_0 - \left[\dfrac{mg}{k}te^{(k/m)t} - \dfrac{m^2g}{k^2}\left(e^{(k/m)t} - 1\right)\right]\mathbf{j}$ and

   $\mathbf{R}(t) = \dfrac{m}{k}\left[1 - e^{-kt/m}\right]\mathbf{v}_0 + \dfrac{mg}{k}\left[\dfrac{m}{k}\left(1 - e^{-kt/m}\right) - t\right]\mathbf{j}$.